Some Logic for Theories of Decisions

Phil 411: Theories of Decision
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In this handout I discuss some basic methods of proof and the meanings of some of the logical connectives you'll see in the class. We'll need this both for working through decision problems, strategic games, and social choice problems and for proving things *about* the models we're developing. Sometimes, for example, we'll want to be able to show that an assumption we're making about the models actually holds or that it makes sense to treat the expected utility of an act as its utility.

Some Basic Definitions

Argument A set of *premises* that are taken to *entail* a *conclusion*.

Validity A valid argument is one in which the truth of the premises guarantees the truth of the conclusion, i.e., it is not possible for the premises to all be true and the conclusion false, i.e., *if* the premises of a valid argument are true, then the conclusion must also be true.

Soundness A sound argument is a valid argument with premises that are known to be true.

Axiom An axiom is a statement that is taken to be self-evidently true or, at least, very widely accepted as such.

Theorem A theorem is a statement for which a proof is given showing that it must be true given other statements we accept as true.

Lemma A lemma is a proposition that we prove in the process of proving a theorem, it is a kind of stepping stone to a larger result, not something we prove for its own sake.

Assumption An assumption is something we merely take to be true for the sake of argument. In order to accept the final conclusion we reach, we need to discharge the assumption.

Basic Proof Methods

Direct Proof We use the method of direct proof when we want to prove that some claim (the conclusion) follows from a set of axioms, definitions, conditions, or theorems we've already proved or that we take to be true (the premises).

A direct proof proceeds by demonstrating how the conclusion follows from the premises using, for example, accepted substitutions (based on the definitions of the terms) and the meanings of the logical connectives (defined below). It may also rely on various accepted rules of inference, e.g., *modus ponens*, *modus tollens*, disjunctive syllogism, etc. For most of the proofs in class, you don't need to know these, but it wouldn't hurt to read a bit about them on Wikipedia.

Conditional Proof We use this method to prove that a conditional, *If p, then q*, is true. We proceed by assuming the truth of the antecedent of the conditional, i.e., the if-clause, and show that the consequent, i.e., the then-clause follows given the axioms, definitions, conditions, or theorems that we already accept. If we want to prove that a bi-conditional, *p if and only if q*, is true, then we must prove it in both directions using conditional proofs, i.e., we must first assume the left-hand side and show that the right-hand side follows, then we must assume the right-hand side and show that the left-hand side follows.

Indirect Proof/Proof by Contradiction/*reductio ad absurdum* We resort to this method of proof when direct proofs are not possible or would be very difficult given the premises we have. This method proceeds by assuming the negation (or opposite) of what we want to prove and demonstrating that a contradiction follows from this assumption, i.e., that we arrive at some claim that contradicts one of our premises. If our assumption leads to a contradiction, then we are justified in inferring the negation of our assumption. Since our assumption was the negation of what we wanted to prove, we get to infer exactly what we set out to prove.

Constructive Proof This is a kind of existence proof or proof by example. We construct a concrete example with the desired properties to show that it exists. We see this, for example, in the proof of the expected utility theorem where we construct a utility function that meets the conditions set out in the theorem to show that it is possible to do so and then proceed to prove that such a function will have various properties.

The Logical Connectives

and, $\wedge$, & 'and' is the logical conjunction. $P \wedge Q$ is true if and only if *both* P and Q are true statements. As such, we can infer either P or Q or both from $P \wedge Q$.

or, $\vee$ 'or' is the logical disjunction. $P \vee Q$ is true if and only if at least one of the disjuncts P or Q is true. Notice that the logical disjunction is *inclusive*, i.e., it does not rule out the possibility that *both* P and Q are true. As such, if we know that $P \vee Q$ and that P is false, then we can infer Q.

not, $\neg$, $\sim$ 'not' is the logical negation. $\neg P$ is true if and only if P is false. As such, if we know that $\neg P$ is false, then we can infer P.

If...,then...., implies, $\supset, \Rightarrow$ This is the logical conditional. The if-clause is called the *antecedent* of the conditional. The then-clause is called the *consequent* of the condtional. $P \Rightarrow Q$ is true if P and Q are both true and if P is false, i.e., if the antecedent is not satisfied. As such, if we know that $P \Rightarrow Q$ is true and that P is true, we can infer Q (this is *modus ponens*). Alternatively, if we know that $P \Rightarrow Q$ is true and that Q is false, then we can infer that P is false or $\neg P$ (this is *modus tollens*).

if and only if, iff, $\iff$ 'iff' is the logical bi-conditional. $P \iff Q$ is true if and only if both P and Q are true or both P and Q are false. As such, if we know the truth-value of either the right-hand side or the left-hand side, then we can infer the truth value of the other.

top, $\top$ This is the symbol for a tautology, i.e., a statement that is true in all possible states of affairs.

bottom, $\bot$ This is the symbol for a contradiction.