Notation for Theories of Decisions

Phil 411: Theories of Decision
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Formal Decision/Game Models

A **formal decision model** $\mathcal{M} = \langle \mathcal{A}, \Omega, \mathcal{P}, \mathcal{O}, g \rangle$ consists of a set of actions $\mathcal{A}$, a set of states Ω, a probability measure over Ω $Pr : 2^{\Omega} \to \mathbb{R}[0,1]$, a set of outcomes $\mathcal{O}$, and a function $g : \mathcal{A} \times \Omega \to \mathcal{O}$ that maps each action $a \in \mathcal{A}$ and state $s \in \Omega$ to an outcome $o \in \mathcal{O}$.

Note that for decisions under ignorance the formal model will not include $\mathcal{P}$.

A formal decision model is derived from a decision problem and underlies any visualization of that problem, like a matrix or tree.

- $\in$ is the symbol for *inclusion in a set.* $s \in \Omega$ can be read either as "s in omega" or "s, an element of omega"
- $g : \mathcal{A} \times \Omega \to \mathcal{O}$ is the notation describing a *function.* In this case, this says that g is a function that maps each ordered pair $< a_x, s_x >$ to an $o \in \mathcal{O}$. We also see this notation used in describing utility functions, e.g., $u : \mathcal{O} \to \mathbb{R}$, which says that u is a function that maps every $o \in \mathcal{O}$ to an a real number.
- $Pr : 2^{\Omega} \to \mathbb{R}[0,1]$ is a probability function that maps every element in the power set of ω to a real number between 0 and 1 (this is because probabilities are represented as real numbers between 0 and 1).
- $\mathbb{R}$ is the symbol for the set of real numbers.
- A *power set* is the set of all possible sets within a set, including the null (or empty) set, e.g., the power set of the set $\{a, b, c\}$ is $\{\emptyset, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}, \{a, b, c\}\}$.
- @ denotes the actual world.

A **strategic game** $\mathcal{G} = \langle \mathcal{N}, \{\mathcal{A}_i\}_{i \in \mathcal{N}}, \mathcal{O}, g, \{\succcurlyeq_i\}_{i \in \mathcal{N}} \rangle$ consists of the following ingredients:

- a finite set of players $\mathcal{N}$
- a set of actions $\mathcal{A}_i$ available to each player $i \in \mathcal{N}$
- a set of outcomes $\mathcal{O}$
- a function $g : \times_{i \in \mathcal{N}} \mathcal{A}_i \to \mathcal{O}$ that maps each action profile $a \in \times_{i \in \mathcal{N}} \mathcal{A}_i$ to an outcome $o \in \mathcal{O}$
- a preference relation $\succcurlyeq_i$ for each player over the set of outcomes $\mathcal{O}$ where $o_1 \succcurlyeq_i o_2$ just in case o_1 is at least as preferred as o_2 by player $i \in \mathcal{N}$

Basic Set Theory Notation

- $\{\}$ is a set, i.e., a collection of elements. Sets can be made up of anything from abstract mathematical objects to tables and chairs.
- $\emptyset$ is the empty set or null set, i.e., the set with no members.
- $|$ means "such that", e.g., $A = \{x|x \in \mathbb{R}, x > 0\}$ says that A is the set of every x such that x is a real number greater than zero.
- $A \cap B$ means the intersection of A and B, i.e., the set of elements that belong to both sets or the overlap of the two sets.
- $A \cup B$ means the union of A and B, i.e., the set of elements that belong to either set or the combination of the two sets.
- $A \subseteq B$ means that A is a subset of B, i.e., that A is included in set B.
- $A \subset B$ means that A is a proper subset of B, i.e., A is a subset of B but is not equal to B.
- 2^A or $\mathcal{P}(A)$ means the power set of A, or the set of all of the subsets of A.
- $A = B$ means that A and B have all of the same elements.
- $a \in A$ means that a is an element of A.
- $a \notin A$ means that a is not an element of A.
- (a, b) is an ordered pair, i.e., a collection of two elements.
- $A \times B$ is the cartesian product of the sets A and B, i.e., the set of all ordered pairs from A and B.
- $\mathbb{N}$ is the set of natural numbers, i.e., the positive integers or counting numbers. This may or may not include zero.
- $\mathbb{R}$ is the set of real numbers, i.e., the set of numbers that includes not only all of the integers but also all of the rational numbers and all of the irrational numbers.

Probability Notation

- $Pr(X)$ is the probability of X. It is a function that takes an outcome or state of affairs as an argument and outputs a real number between 0 and 1.
- $Pr(\overline{X})$ means the probably of not-X.
- $Pr(X \cap Y)$ means the probability of X intersection Y, i.e., the probability that something is in both sets.
- $Pr(X \cup Y)$ means the probability of X union Y, i.e., the probability that something is in one or both sets.
- $Pr(X|Y) = \frac{Pr(X \cap Y)}{Pr(Y)}$ is the *conditional probability of* X *given* Y, i.e., the probability that X will be the case given that Y is the case.

Utility Notation

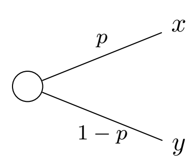
- $u(x)$ is the utility of x. This represents a utility function that maps the argument to a real number.
- $u(x) > u(y)$ says that the utility of x is strictly greater than the utility of y.
- $u(x) < u(y)$ says that the utility of x is strictly less than the utility of y.
- $u(x) \geq u(y)$ says that the utility of x is greater than or equal to the utility of y.
- $u(x) \leq u(y)$ says that the utility of x is less than or equal to the utility of y.
- $u(x) = u(y)$ says that the utility of x and y are the same.
- These relationships can each be defined in terms of the others.
  - $x > y$ iff $x \nleq y$
  - $x < y$ iff $x \ngeq y$
  - $x = y$ iff $x \not< y$ and $x \not> y$ iff $x \leq y$ and $x \geq y$

Preference Notation

- $o_1 \succ o_2$ designates that o_1 is preferred to o_2.
- $o_1 \sim o_2$ designates that o_1 and o_2 are preferred equally.
- $o_1 \succcurlyeq o_2$ designates that o_1 is at least as preferred as o_2.
- Again, these relations are interdefinable:
  - $o_1 \succ o_2$ iff $o_1 \succcurlyeq o_2$ and $o_1 \nsim o_2$.
  - $o_1 \sim o_2$ iff $o_1 \succcurlyeq o_2$ and $o_2 \succcurlyeq o_1$.
  - $o_1 \succcurlyeq o_2$ iff $o_1 \succ o_2$ or $o_1 \sim o_2$.

Lottery Notation

- $L(p, x, y)$ designates the *lottery* that gives x with probability p and gives y with probability $1 - p$.



- The set of *lotteries* $\mathfrak{L}$ is built up as follows:
  - $o \in \mathfrak{L}$ for each outcome $o \in \mathcal{O}$, i.e., each outcome is itself a lottery.
  - If $L_1 \in \mathfrak{L}$ and $L_2 \in \mathfrak{L}$, then $L(p, L_1, L_2) \in \mathfrak{L}$ for any $p \in \mathbb{R}[0, 1]$, i.e., a new lottery can be constructed out of any of the lotteries in $\mathfrak{L}$.
  - Nothing else is in $\mathfrak{L}$.