Decision Theory

1.5 Paradoxes and Problems

George Mason University, Spring 2019

Here's the question we've been thinking about:
When making a decision under risk, *why* maximize expected utility?

Here's an answer we considered (call it the *law of large numbers* reply):

Answer. In the long run, you will be better off by maximizing EU.

Reply. No real-life decision maker will ever face a decision an infinite number of times. Keynes: "In the long run we are all dead."

Reply. Many decisions are unique, such as the decision to marry a particular partner, the decision to start a particular war, and so on.

So we turned to an *indirect axiomatic argument* to try to show that we should maximize expected utility.

- The aim was to show that for any action $u(a) = EU(a)$.
- We did this by adopting Ax1-Ax7 and proving the Expected Utility Theorem from them.

Recall the Expected Utility Theorem:

Thm 1.3.7. If an agent's preferences over lotteries satisfy Ax1-Ax7, then there is a utility function $u : \mathfrak{L} \to \mathbb{R}[0, 1]$ such that:

(i) $u(L_1) > u(L_2)$ if and only if $L_1 \succ L_2$.

(ii) $u(L(p, L_1, L_2)) = p \times u(L_1) + (1 - p) \times u(L_2)$.

(iii) Any u' satisfying (i) and (ii) is a positive linear transformation of u.

Notice that each act $a \in \mathcal{A}$ is itself a lottery whose expected utility is its utility. We forged a connection between utility and expected utility. Resnik: "In choosing an act whose expected utility is maximal an agent is simply doing what he wants to do!"

Now we face some new problems:

Reply. It now seems that an agent whose preferences satisfy Ax1-Ax7 doesn't even need decision theory.

Counter-reply. The maxim 'Maximize EU!' should be understood as 'Have preferences that satisfy the structural constraints Ax1-Ax7 (and then just do what you most prefer to do)!'

This is a common way of understanding what decision theory is really about. It tells us how we ought to structure our preferences if we are to be rational, but once we've done that, it has nothing else to tell us. In that case, we'll just do the action that has the highest EU because it has the highest utility for us. The model, then, is just a way of bringing out how a rational agent would behave. It doesn't itself help us make decisions.

Even on this understanding, there are some sophisticated problems we need to confront. We're going to look at some paradoxes in which expected utility seems to lead us astray and at some responses to those paradoxes.

Ex. Allais Paradox.
(Due to Maurice Allais in 1953.)

You are given a choice between the following two payouts:

A: \$1M.

B: You receive \$5M with 10% probability, \$1M with 89% probability, and nothing with 1% probability.

Do you choose A or B?

Give it some thought and write down your choice.

Next you are next given a choice between the following two payouts:

C: You receive \$5M with 10% probability and nothing with 90% probability.

D: You receive \$1M with 11% probability and nothing with 89% probability.

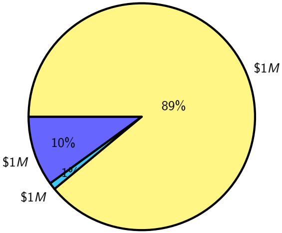
Do you choose C or D?

Give it some thought and write down your choice.

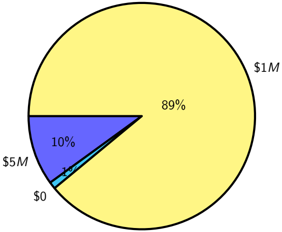
Now consider this way of rewriting the choices as lotteries:

Do you choose A or B:

A: $L(0.11, \$1M, \$1M)$

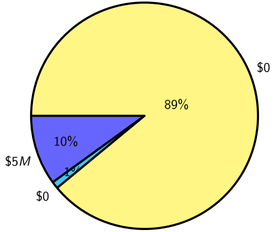


B: $L(0.11, L(\frac{10}{11}, \$5M, \$0), \$1M)$

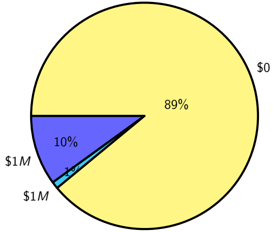


Do you choose C or D:

C: $L(0.11, L(\frac{10}{11}, \$5M, \$0), \$0)$



D: $L(0.11, \$1M, \$0)$



If you chose:

A: \$1M or $L(0.11, \$1M, \$1M)$
and
C: You receive \$5M with 10% probability and nothing with 90% probability or $L(0.11, L(\frac{10}{11}, \$5M, \$0), \$0)$

or

B: You receive \$5M with 10% probability, \$1M with 89% probability, and nothing with 1% probability or $L(0.11, L(\frac{10}{11}, \$5M, \$0), \$1M)$
and
D: You receive \$1M with 11% probability and nothing with 89% probability or $L(0.11, \$1M, \$0)$

then your preferences cannot be represented by a utility function because you violate Better Prizes. If you think $u(A) > u(B)$ then, by Better Prizes, you should think $u(D) > u(C)$. *You must think that the $\$1M$ sure thing is a better prize than the $\frac{10}{11}$ chance at $\$5M$.* The problem is, Allais showed that most people don't think this, and most people think their preferences are reasonable even on reflection.

Let's look at it another way:

$EU(A) = 1 \times u(\$1\text{M})$.

$EU(B) = 0.1 \times u(\$5\text{M}) + 0.89 \times u(\$1\text{M}) + 0.01 \times u(\$0\text{M})$.

$EU(C) = 0.1 \times u(\$5\text{M}) + 0.9 \times u(\$0\text{M})$.

$EU(D) = 0.11 \times u(\$1\text{M}) + 0.89 \times u(\$0\text{M})$.

$EU(A) - EU(B) = 0.11 \times u(\$1\text{M}) - 0.1 \times u(\$5\text{M}) - 0.01 \times u(\$0\text{M})$.

$EU(D) - EU(C) = 0.11 \times u(\$1\text{M}) - 0.1 \times u(\$5\text{M}) - 0.01 \times u(\$0\text{M})$.

If you choose A over B, then presumably
$EU(A) - EU(B) = EU(D) - EU(C) > 0$.

So if you are an EU-maximizer, then you choose D over C.

Reply. The outcomes shouldn't be specified solely in terms of money. The outcome in offer B is $0 *plus serious disappointment*.

Counter-reply. Every objection to the Principle of Maximizing EU might be thwarted by fiddling with the outcomes.

Reply. Bite the bullet. Leonard "Jim" Savage: An agent who chooses A and C is irrational because they violate the *sure-thing principle*.

Sure Thing Principle. "[Let f and g be any two acts], if a person prefers f to g, either knowing that the event B obtained, or knowing that the event not-B obtained, then he should prefer f to g even if he knows nothing about B."

Imagine a raffle with a hundred tickets:

	Ticket 1	Tickets 2-11	Tickets 12-100
gamble A	\$1M	\$1M	\$1M
gamble B	\$0M	\$5M	\$1M
gamble C	\$0M	\$5M	\$0M
gamble D	\$1M	\$1M	\$0M

The third column should be ignored when deciding between the gambles.

Counter-reply. Why satisfy the sure-thing principle?

The Allais Paradox plays on the common preference for a good for certain over a risky chance for a more valuable good.

As such, it challenges Ax5 (Better Prizes) and strikes at the heart of EU theory. Perhaps it is *not* irrational to sometimes prefer a certain outcome to a risky outcome even if the risky outcome offers a better prize.

The Allais Paradox is one of the problems that motivates theories that build in some way of accounting for risk aversion.

Ex. Ellsberg Paradox.

An urn contains 90 balls. You know that 30 of these are yellow. You also know that the remaining 60 balls are either red or blue, but you do not know the proportion. I am about to draw a ball from the urn and I give you a choice between the following two payouts:

A: You receive \$100 if a yellow ball is drawn and \$0 otherwise.

B: You receive \$100 if a red ball is drawn and \$0 otherwise.

Do you choose A or B?

Suppose that I had instead offered a choice between the following two payouts:

C: You receive \$100 if either a red or blue ball is drawn and \$0 otherwise.

D: You receive \$100 if either a yellow or blue ball is drawn and \$0 otherwise.

Do you choose C or D?

Suppose that you assign a conditional probability of p to getting a red ball given that you get a red or blue ball.

$EU(A) = \frac{1}{3} \times u(\$100) + \frac{2}{3} \times u(\$0)$.

$EU(B) = \frac{1}{3} \times u(\$0) + \frac{2}{3} \times p \times u(\$100) + \frac{2}{3} \times (1 - p) \times u(\$0)$.

$EU(C) = \frac{1}{3} \times u(\$0) + \frac{2}{3} \times u(\$100)$.

$EU(D) = \frac{1}{3} \times u(\$100) + \frac{2}{3} \times p \times u(\$0) + \frac{2}{3} \times (1 - p) \times u(\$100)$.

$EU(A) - EU(B) = \frac{1-2p}{3} \times u(\$100) + \frac{2p-1}{3} \times u(\$0)$.

$EU(D) - EU(C) = \frac{1-2p}{3} \times u(\$100) + \frac{2p-1}{3} \times u(\$0)$.

If you choose A over B, then presumably
$EU(A) - EU(B) = EU(D) - EU(C) > 0$.

So if you are an EU-maximizer, then you choose D over C.

Reply. Bite the bullet. An agent who chooses A and C is irrational because they violate Better Chances (and the sure-thing principle).

	$[\frac{1}{3}]$ Yellow	$[\frac{2}{3} \times p]$ Red	$[\frac{2}{3} \times (1 - p)]$ Blue
gamble A	\$100	\$0	\$0
gamble B	\$0	\$100	\$0
gamble C	\$0	\$100	\$100
gamble D	\$100	\$0	\$100

If you choose A over B, then, presumably, you think that $p < 0.5$. If you thought that $p > 0.5$, you would choose B because you would think that you had a better chance of getting a Red ball than a Yellow or Blue one.

But, if you think $p < 0.5$, you should choose D over C because you must think that there's a greater than $\frac{2}{3}$ chance of getting a Yellow or Blue ball.

Counter-reply. Why satisfy Better Chances?

Reply. This is a decision under ignorance so the Principle of Maximizing EU does not apply.

Counter-reply. This reply is not open to subjectivists about probability who think that there are no real decisions under ignorance.

The Ellsberg Paradox plays on the common preference for known risks over unknown risks.

It's not exactly clear why such a preference should be counted irrational. Ellsberg, in his dissertation, argued that decisions under uncertainty or ambiguity (such as this) generally may not be in line with well defined subjective probabilities. Perhaps this isn't a problem for the preferences but for expected utility theory itself.

Preferences can only be represented by a utility curve if they satisfy Ax1-Ax7, but perhaps Ax6 (Better Chances) is not as uncontroversial as it had originally seemed to us. The upshot, then, is another serious challenge to EU theory. A vast literature exists trying to meet this challenge.

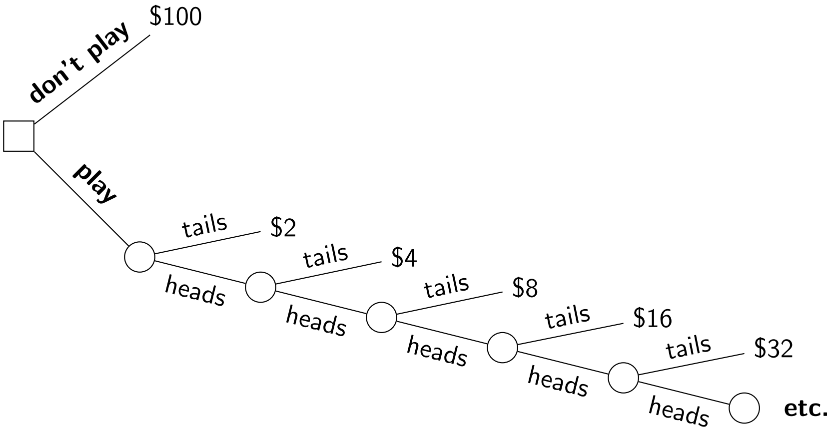
Ex. St. Petersburg Paradox.
(Due to Nicolaus Bernoulli in 1713, but named for the solution published by his cousin Daniel Bernoulli in the *Commentaries of the Imperial Academy of Science of Saint Petersburg* in 1738.)

You are given a choice between the following two payouts:

A: \$100.

B: A fair coin is flipped until it lands tails. If the coin lands tails on the first toss, then you receive \$2. If the coin lands tails on the second toss, then you receive \$4. In general, if the coin lands tails on the nth toss, then you receive $\$2^n$.

Do you choose A or B?



Let $EMV(a)$ designate the *expected monetary value* of $a \in \mathcal{A}$.

$EMV(A) = \$100$.

$EMV(B) = \frac{1}{2} \times \$2 + \frac{1}{4} \times \$4 + \frac{1}{8} \times \$8 + ... = \$1 + \$1 + \$1 + ... = \$\infty$.

Reply. $EMV \neq EU$, money has diminishing marginal utility. This was D. Bernoulli's solution.

Counter-reply. The St. Petersburg Paradox can be reframed in terms of utilities. If the coin lands tails on the first toss, then you receive a prize worth 2 utiles, and so forth.

Reply. There is, or should be, an upper bound on utility.

Counter-reply. This upper bound is ad hoc.

Reply. Richard Jeffrey (1983): "Put briefly and crudely, our rebuttal of the St. Petersburg paradox consists in the remark that anyone who offers to let the agent play the St. Petersburg game is a liar, for he is pretending to have an indefinitely large bank."

Jeffrey's basic idea is that in thinking about a decision an agent assigns preferences only over propositions to which she assigns some positive degree of belief, i.e., over ones she thinks are actually possible. But no rational agent would believe that any mortal person or institution can pay off arbitrarily large prizes. The St. Petersburg game, as such, doesn't even get assigned a preference.

Counter-reply. All sorts of hypothetical prizes can be allowed.

Ex. Moscow Game.

You are given a choice between the following two payouts:

A: A fair coin is flipped until it lands tails. If the coin lands tails on the first toss, then you receive \$2. If the coin lands tails on the second toss, then you receive \$4. In general, if the coin lands tails on the nth toss, then you receive $\$2^n$.

B: A biased coin that lands tails with probability 0.4 is flipped until it lands tails. If the coin lands tails on the first toss, then you receive \$2. If the coin lands tails on the second toss, then you receive \$4. In general, if the coin lands tails on the nth toss, then you receive $\$2^n$.

Do you choose A or B?

Intuitively, B is preferable.

However, $EMV(A) = EMV(B) = \infty$.

The probability that the coin lands tails seems like it should matter, but the *EMV* of each gamble is the same no matter what probability we assign.

Reply. One proposed solution is that the appropriate comparison is not EU but **Relative EU**, i.e., compare the EMV at each toss. In this case, it seems clear that it's better to play A than B in the Moscow game. This is counterintuitive, since it seems you'd want to play the game that gives you a better chance of more tosses, but let's roll with it for a minute.

But then consider...

Ex. Leningrad Game.

You are given a choice between the following two payouts:

A: A fair coin is flipped until it lands tails. If the coin lands tails on the first toss, then you receive \$2. If the coin lands tails on the second toss, then you receive \$4. In general, if the coin lands tails on the nth toss, then you receive $\$2^n$.

B: A biased coin that lands tails with probability 0.4 is flipped until it lands tails. If the coin lands tails on the first toss, then you receive \$2. If the coin lands tails on the second toss, then you receive \$4. In general, if the coin lands tails on the nth toss, then you receive $\$2^n$. However, if the coin lands tails on the third toss, then you receive \$8 *and* get to play the St. Petersburg Game (which is just the game in A).

This hijacks the proposed solution to the Moscow game. On the 4th toss, B has an infinite EMV whereas A has an $EMV = \frac{1}{8} \times \$8 = \$1$.

Ex. Two Envelope Paradox.

You are offered a choice between two envelopes A and B. You know that one of these envelopes contains twice as much money as the other, but you do not know how much money is in either envelope. You pick A. But right before you open this envelope, you are offered the opportunity to switch to B. Do you switch?

If you are an EU-Maximizer, then it seems that you should switch.

Suppose that there is $\$S$ in envelope A.

$EMV(\text{A}) = \$S.$

$EMV(\text{B}) = \frac{1}{2} \times \$\frac{1}{2}S + \frac{1}{2} \times \$2S = \$\frac{5}{4}S.$

$EMV(\text{B}) > EMV(\text{A}).$

But now suppose I offer you the opportunity to switch back...

Reply. The Two Envelope Paradox requires that there is an infinite amount of money in the world. If there is only $\$T$ available, then the envelope with more money can contain no more than $\$\frac{2}{3}T$. If this envelope contains $\$\frac{2}{3}T$, then the other envelope must contain $\$\frac{1}{3}T$.

Counter-reply. We can work with utilities instead.